Central Limit Theorem

Here is a look at a strong assumptions version of the CLT. The CLT works under weaker assumptions, but its proof is more difficult.

Theorem 1 *If $X_1,\ X_2,\ldots$ are independent and identically distributed like a random variable X with finite mean, μ, and finite variance σ^2, then*

$$\lim_{n\to\infty} \mathrm{P}\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq t\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-u^2/2}du = \Phi(t) \tag{1}$$

In other words, $(S_n - n\mu)/(\sigma\sqrt{n}) \xrightarrow{d} Z$ or $(\bar{X} - \mu)/(\sigma/\sqrt{n}) \xrightarrow{d} Z$ where $Z \sim N(0,1)$.

Proof 1 *Let*

$$Z_n = \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma\sqrt{n}} \tag{2}$$

Since $(X_i - \mu)/(\sigma\sqrt{n})$, $i = 1, 2, \ldots$ are independent random variables, and since the mgf of the sum of independent random variables is equal to the product of their mgf's, we can write

$$\begin{aligned} M_{Z_n}(s) &= \prod_{i=1}^{n} \mathrm{E}\left[\exp\left(s\frac{(X_i - \mu)}{\sigma\sqrt{n}}\right)\right] && (3)\\ &= \left(\mathrm{E}\left[e^{(s/(\sigma\sqrt{n}))(X-\mu)}\right]\right)^n && (4)\end{aligned}$$

since the random variables are iid. Hence

$$M_{Z_n}(s) = \left[M_{X-\mu}\left(\frac{s}{\sigma\sqrt{n}}\right)\right]^n \tag{5}$$

Now from before [see mgf notes] we know that

$$M_{X-\mu}(s) = 1 + s\mathrm{E}(X-\mu) + \frac{s^2}{2!}\mathrm{E}\left[(X-\mu)^2\right] + \cdots + \frac{s^r}{r!}\mathrm{E}\left[(X-\mu)^r\right] + \cdots \tag{6}$$

and since $\mathrm{E}(X-\mu) = 0$ and $\mathrm{E}\left[(X-\mu)^2\right] = \sigma^2$ we have

$$M_{X-\mu}(s) = 1 + \sigma^2\frac{s^2}{2!} + \cdots + \mathrm{E}\left[(X-\mu)^r\right]\frac{s^r}{r!} + \cdots \tag{7}$$

Thus,

$$M_{X-\mu}\left(\frac{s}{\sigma\sqrt{n}}\right) = 1 + \sigma^2\frac{s^2}{2n\sigma^2} + \theta_n\frac{s^2}{2n\sigma^2} \tag{8}$$

where $\theta_n \to 0$ as $n \to \infty$. So

$$\begin{aligned} M_{Z_n}(s) &= \left(1 + \frac{s^2}{2n} + \frac{\theta_n s^2}{2n\sigma^2}\right)^n && (9)\\ &= \left(1 + \frac{\frac{s^2}{2} + \frac{\theta_n}{2}\cdot\frac{s^2}{\sigma^2}}{n}\right)^n && (10)\end{aligned}$$

Now let $c_n = \frac{\theta_n s^2}{2\sigma^2}$ and recall that

$$\lim_{n\to\infty}\left(1 + \frac{a + c_n}{n}\right)^n = e^a$$

if $\lim_{n\to\infty} c_n = 0$. We now have

$$\lim_{n\to\infty} M_{Z_n}(s) = e^{s^2/2} \tag{11}$$

But, $e^{s^2/2}$ is the mgf of a standard normal random variable.